Vibration Characteristics of Cantilevered Thick Cylindrical Shallow Shells

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I. Introduction

THE analysis of free vibration of shells has received widespread attention in recent years because of the practical importance in structural, mechanical, and aerospace applications. A relatively complete study by Leissa[1] summarized most of the related work on shell analysis before 1973. Much research has been conducted since that time.[2,3] Recently, the authors reviewed some of the outstanding works in this field.[4,5] Publications dealing with the vibration response of thick shallow shells have been less common. One of the excellent accounts in this area was presented by Palazotto and Linnemann[6] on vibration and buckling of composite cylindrical panels. In this Note, the authors attempt to extend their earlier work[4,5] to study the free vibration of thick rectangular shallow shells.

In some practical cases, the constitutive panels of structures must be of considerable thickness. In such cases, the thin-shell theory underestimates deflection and overestimate frequency and buckling load. Thus, a numerical model that represents more accurately the thickness effects of the constitutive panels is necessary. In this study, a higher-order theory incorporating the effects of rotary inertia and shear deformation, and allowing a cubic shear distribution through the thickness, is developed.

This Note also aims to generalize the Ritz procedures from thin plates and shallow shells[4,5,7] to thick shallow shells. An additional rotation field is required, and the freedom of motion increases from a 3-deg to a 5-deg system. The governing eigenvalue equation is derived from a displacement-based energy functional. Numerical vibration frequencies for thick cantilevered cylindrical shallow shells are presented.

II. Theory and Formulation

Consider a homogeneous, isotropic, cantilevered thick shallow cylindrical shell of rectangular planform with length a, width b, and thickness h. The chordwise principal radius of curvature of the midsurface of shell ($z = 0$) is denoted by R_y.

The normal and shear strain fields for the shell described above are

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{1a}$$

$$\epsilon_y = \frac{1}{1 + z/R_y}\left(\frac{\partial v}{\partial y} + \frac{w}{R_y}\right) \tag{1b}$$

$$\epsilon_z = 0 \tag{1c}$$

$$\gamma_{yz} = \frac{1}{1 + z/R_y}\left(\frac{\partial w}{\partial y} - \frac{v}{R_y}\right) + \frac{\partial v}{\partial z} \tag{1d}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{1e}$$

$$\gamma_{xz} = \frac{1}{1 + z/R_y}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{1f}$$

In Eq. (1d), we assume transverse inextensibility. The strain expressions reduce to flat-plate equations if the Lamé parameter $1/(1 + z/R_y)$ is equivalent to unity ($R_y \to \infty$).

The formulation assumes a higher-order shear deformable displacement field:

$$u = u_o + z\theta_1^u + z^2\theta_2^u + z^3\theta_3^u \tag{2a}$$

$$v = (1 + (z/R_y))v_o + z\theta_1^v + z^2\theta_2^v + z^3\theta_3^v \tag{2b}$$

$$w = w_o \tag{2c}$$

where (u_o, v_o, w_o) are the displacements at the midsurface. The rotations of a point are θ_1^u and θ_1^v, and the rotation parameters are θ_i^u and θ_i^v ($i = 2, 3$).

Substituting u, v, and w from Eqs. (2a–2c) into Eqs. (1a–1f) and enforcing zero shear strains at the top and bottom surfaces of the shell, the rotation parameters θ_2^u, θ_2^v, θ_3^u, and θ_3^v can be solved to obtain the following displacement field:

$$u = u_o + z\theta_1^u - \frac{4z^3}{3h^2}\left(\frac{\partial w_o}{\partial x} + \theta_1^u\right) \tag{3a}$$

$$v = v_o + z\left(\frac{v_o}{R_y} + \theta_1^v\right) + \frac{z^2}{3R_y}\left(\frac{\partial w_o}{\partial y} + \theta_1^v\right) - \frac{4z^3}{3h^2}\left(\frac{\partial w_o}{\partial y} + \theta_1^v\right) \tag{3b}$$

$$w = w_o \tag{3c}$$

where terms of $\mathcal{O}(1/R_y^2)$ are dropped in view of the shallowness of the shell. Using the above displacement field, the strain components in Eqs. (1a–1f) in which the transverse shear strain γ_{yz} is a noneven, cubic expression of the thickness coordinate z can be obtained.[8]

Assuming no normal stress, the strain energy of a thick shallow shell is

$$\mathcal{U} = \frac{E}{2(1 - \nu^2)}\iiint_V \left[\epsilon_x^2 + 2\nu\epsilon_x\epsilon_y + \epsilon_y^2 + \frac{1 - \nu}{2}\left(\gamma_{yz}^2 + \gamma_{xz}^2 + \gamma_{xy}^2\right)\right]\mathrm{d}V \tag{4}$$

and the kinetic energy is

$$\mathcal{T} = \frac{\rho}{2}\iiint_V \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}V \tag{5}$$

where V is the volume of shell and ρ is the mass density per unit volume. The displacement and rotation amplitude functions can be approximated as

$$U_o(x, y) = \sum_{i=1}^{m} c_i^u \phi_i^u(x, y) \tag{6a}$$

$$V_o(x, y) = \sum_{i=1}^{m} c_i^v \phi_i^v(x, y) \tag{6b}$$

$$W_o(x, y) = \sum_{i=1}^{m} c_i^w \phi_i^w(x, y) \tag{6c}$$

$$\Theta_1^u(x, y) = \sum_{i=1}^{m} \bar{c}_i^u \Phi_i^u(x, y) \tag{6d}$$

$$\Theta_1^v(x, y) = \sum_{i=1}^{m} \bar{c}_i^v \Phi_i^v(x, y) \tag{6e}$$

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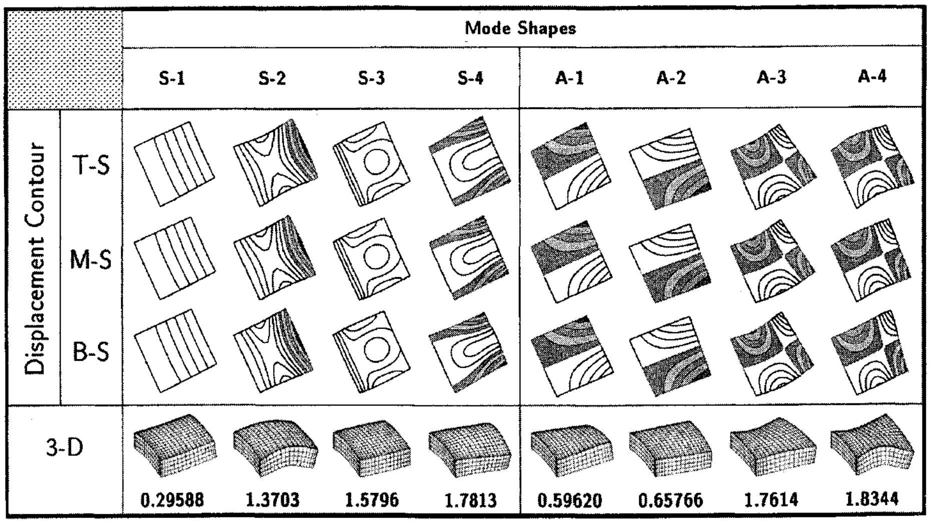

Fig. 1 *W*-contour (in-plane displacement incorporated) and three-dimensional vibration mode shapes of a CFFF (cantilever or one clamped edge and the other free) thick cylindrical shallow shell with ν = 0.3, b/R_y = 0.5, h/b = 0.3, and a/b = 1.0 (T-S, top surface; M-S, midsurface; and B-S, bottom surface).

in which c_i^u, c_i^v, c_i^w, $\bar{c}_i^u$, $\bar{c}_i^v$ are the unknown coefficients and ϕ_i^u, ϕ_i^v, ϕ_i^w, Φ_i^u, Φ_i^v are the corresponding shape functions.[8]

Minimization of the energy functional

$$\Pi = \mathcal{U}_{\max} - \mathcal{T}_{\max} \tag{7}$$

with respect to the coefficients, according to the Ritz procedure, yields

$$(K - \lambda^2 M)\{C\} = \{0\}; \qquad \lambda = \omega ab\sqrt{\rho h/D} \tag{8}$$

with stiffness matrix K, mass matrix M, and eigenvalue λ.

III. Results and Discussion

A convergence study was carried out where the number of terms m of the shape functions was increased from 45 to 78. The corresponding determinant sizes are from 45 × 5 to 78 × 5. It was observed that 78 terms is adequate to ensure satisfactory convergence of eigenvalues.

The frequency response of cantilevered thick cylindrical shallow shells with aspect ratios, a/b = 1.0 and 3.0 is presented in Table 1. The shallowness ratio b/R_y varies from 0.0 (a flat plate) to 0.5 (a shallow shell), whereas the thickness ratio h/b ranges from 0.1 (a moderately thick shell) to 0.5 (a thick shell).

The vibration frequencies can be classified into two distinct groups according to their vibration mode shapes with respect to the xz plane. They are the symmetric (S) and antisymmetric (A) modes. The first two frequencies for each class are presented. The effects of h/b, b/R_y, and a/b can be observed. The frequency parameter $\lambda' = \omega a\sqrt{(\rho/E)}$ increases for higher h/b, indicating that the frequency of a thicker shell is higher than that of a thinner shell. This is generally expected because the shell structural stiffness increases with thickness. For a thin shallow shell (h/b = 0.01 or less), increases in the shallowness ratio induce higher shell stiffness, as discussed at length by Lim and Liew.[5] However, as the shell gets thicker (h/b = 0.1, 0.3, 0.5), the effect of b/R_y on λ' is less significant.

To examine the effect of a/b, it is convenient to note that changing a/b causes changes in b while keeping a constant to ensure its independence of λ' and ω. As observed, an increase in a/b reduces the lowest frequencies for the two classes (S-1 and A-1). In other words, a longer shell has lower fundamental frequencies. For higher modes, the effect of a/b is rather obscure and its role depends on the shell thickness.

A set of vibration mode shapes in W-contour (in-plane displacement incorporated) and three-dimensional plots is presented in Fig. 1. The modes are again divided into symmetric and antisymmetric. The contour plots for the top, middle, and bottom surfaces are presented, and the lines of demarcation in these contour plots are the nodal lines. We can classify the mode shapes into spanwise bending (SB), chordwise bending (CB), twisting (T), pure shear (PS) modes or a combination of them. For a/b = 1.0, the various modes for S-1 to S-4 are, respectively, 1-SB, 2-SB, 1-PS, 1-CB. The A-1 mode is 1-T. The A-2 to A-4 modes are combinations of T and PS modes.

Table 1 Frequency parameter $\lambda' = \omega a\sqrt{(\rho/E)}$ for a cantilevered thick cylindrical shallow shell with ν = 0.3

			Mode sequence number			
			Symmetric		Antisymmetric	
a/b	b/R_y	h/b	S-1	S-2	A-1	A-2
1	0.0	0.1	0.10384	0.60840	0.24407	0.65833
		0.3	0.29215	1.3693	0.59950	0.65833
		0.5	0.44190	1.5797	0.65833	0.79217
	0.1	0.1	0.10450	0.60961	0.24411	0.65833
		0.3	0.29233	1.3693	0.59938	0.65830
		0.5	0.44200	1.5797	0.65790	0.79215
	0.3	0.1	0.10951	0.61898	0.24439	0.65830
		0.3	0.29371	1.3697	0.59841	0.65810
		0.5	0.44256	1.5796	0.65450	0.79184
	0.5	0.1	0.11853	0.63641	0.24487	0.65826
		0.3	0.29588	1.3703	0.59620	0.65766
		0.5	0.44251	1.5794	0.64768	0.79049
3	0.0	0.1	0.034385	0.21380	0.20486	0.31375
		0.3	0.10199	0.61032	0.31375	0.53772
		0.5	0.16731	0.93707	0.31375	0.74957
	0.1	0.1	0.034710	0.21507	0.20483	0.31371
		0.3	0.10223	0.61020	0.31298	0.53777
		0.5	0.16760	0.93635	0.31179	0.74949
	0.3	0.1	0.036381	0.22474	0.20450	0.31347
		0.3	0.10129	0.60880	0.30683	0.53753
		0.5	0.16466	0.92970	0.29546	0.74767
	0.5	0.1	0.035298	0.24200	0.20300	0.31297
		0.3	0.082285	0.60395	0.29405	0.53400
		0.5	0.12580	0.91184	0.25818	0.73808

IV. Concluding Remarks

An extension of the authors' earlier work to the free vibration analysis for shear-deformable cantilevered thick cylindrical shallow shells of rectangular planform is presented. The Ritz method is employed, using a higher-order shear deformation theory with transverse shear deformation and rotary inertia. Downward convergence of eigenvalues implying the existence of upperbound eigenvalues is demonstrated. As expected, the results show that the shell stiffness and frequency increase for a thicker shell. Increasing the shallowness ratio results in higher frequency for a thin shell, but the effect is insignificant for a thick shell.

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Relationship of Anisotropic and Isotropic Materials for Antiplane Problems

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Introduction

THE problem of determining the stress distribution around defects has been considered by many authors over the past years. The strength of materials is influenced by the existing defects such as cracks, which can cause the stress concentration near the defects. The problem of finding the stress singularities at the apex of an isotropic elastic wedge was considered by Williams,[1] who used the eigenfunction-expansion method. Tranter[2] used the Mellin transform in conjunction with the Airy stress function representation of plane elasticity to solve for the isotropic wedge problem. Stroh[3] obtained an analytical solution to the problem of a plane crack in an anisotropic material of infinite extent. Following the approach of Stroh, Ting[4] studied the stress distribution near the composite wedge of anisotropic materials. A complex function representation of a generalized Mellin transform was employed by Bogy[5] for analyzing stress singularities in an anisotropic wedge. The antiplane problem of two dissimilar anisotropic wedges of arbitrary angles that are bonded together perfectly along a common edge has been considered recently by Ma and Hour.[6] They found the order of the stress singularity to be always real for the antiplane dissimilar anisotropic wedge problems. That is quite a different characteristic from the in-plane case in which the complex type of stress singularity might exist.

A previous analysis of the dissimilar anisotropic antiplane wedge problem by Ma[7] showed that, if an effective angle and an effective material constant are introduced for the anisotropic case, then the order of singularity for the anisotropic material can be obtained from the result of the isotropic case. The results obtained by Ma and Hour[6] have been extended to the present study for the full field, and the correspondence in the stress-displacement relationship has been established between anisotropic and isotropic materials. The reduction in the number of elastic constants considerably simplifies the description of the stress and displacement state. The Mellin transform method, originally applied to the analysis of wedge problems by Tranter,[2] is used for analyzing the antiplane problem of anisotropic material. The material symmetry is assumed to be such that the in-plane and the antiplane deformations are uncoupled. A correspondence is obtained between the anisotropic and the isotropic problem. Through such a correspondence, the relationship of the stresses and displacement for the anisotropic and the corresponding isotropic problem is established for both polar and Cartesian coordinate systems. Any solution of anisotropic material for the antiplane problem can be obtained by solving a corresponding isotropic problem.

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General Solutions in Mellin-Transform Domain

Isotropic Case

The isotropic material is considered first. It is well known that the only nonvanishing displacement component w^i is along z axis for the antiplane deformation. The equilibrium equation for the nonvanishing displacement w^i is given by the following partial differential equation:

$$\frac{\partial^2 w^i}{\partial r^2} + \frac{1}{r}\frac{\partial w^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w^i}{\partial \theta^2} = 0 \tag{1}$$

The nonvanishing stresses are

$$\tau^i_{rz} = \mu \frac{\partial w^i}{\partial r} \tag{2}$$

$$\tau^i_{\theta z} = \frac{\mu}{r}\frac{\partial w^i}{\partial \theta} \tag{3}$$

Let the Mellin transform of a function $f(r)$ be denoted by $\hat{f}(s)$:

$$\hat{f}(s) = M\{f; s\} = \int_0^\infty f(r) r^{s-1}\,\mathrm{d}r \tag{4}$$

where s is a complex transform parameter. Let $\hat{w}^i(s,\theta)$, $\hat{\tau}^i_{rz}(s,\theta)$, $\hat{\tau}^i_{\theta z}(s,\theta)$ in this order denote the Mellin transforms of $w^i(r,\theta)$, $r\tau^i_{rz}(r,\theta)$, and $r\tau^i_{\theta z}(r,\theta)$ with respect to r. Accordingly,

$$\hat{w}^i(s,\theta) = \int_0^\infty w^i(r,\theta) r^{s-1}\,\mathrm{d}r \tag{5}$$

$$\hat{\tau}^i_{rz}(s,\theta) = \int_0^\infty \tau^i_{rz}(r,\theta) r^{s}\,\mathrm{d}r \tag{6}$$

$$\hat{\tau}^i_{\theta z}(s,\theta) = \int_0^\infty \tau^i_{\theta z}(r,\theta) r^{s}\,\mathrm{d}r \tag{7}$$

Application of the inversion formula to Eqs. (5–7) gives

$$w^i(r,\theta) = \frac{1}{2\pi i}\int_{\rho - i\infty}^{\rho + i\infty} \hat{w}^i(s,\theta) r^{-s}\,\mathrm{d}s \tag{8}$$

$$\tau^i_{rz}(r,\theta) = \frac{1}{2\pi i}\int_{\rho - i\infty}^{\rho + i\infty} \hat{\tau}^i_{rz}(s,\theta) r^{-s-1}\,\mathrm{d}s \tag{9}$$

$$\tau^i_{\theta z}(r,\theta) = \frac{1}{2\pi i}\int_{\rho - i\infty}^{\rho + i\infty} \hat{\tau}^i_{\theta z}(s,\theta) r^{-s-1}\,\mathrm{d}s \tag{10}$$

Applying the Mellin transform (5) to Eq. (1) yields an ordinary differential equation for $\hat{w}^i$, the general solution of which is

$$\hat{w}^i(s,\theta) = a(s)\sin(s\theta) + b(s)\cos(s\theta) \tag{11}$$